The Application of the Spectral Envelope in the Business Cycle

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Abstract

Progress in the theory of multivariate time series analysis has taken place in two contexts: the time and frequency domains. Several methods have been proposed but, depending on the data, it is sometimes necessary to apply appropriate transformations to reduce the variance of the trends or the estimates. In the case of multivariate series, it has been shown that the use of a spectral envelope facilitates the generation of a good estimate by using principal components in frequency space. To illustrate the usefulness of the spectral envelope, we use the relationship we find between business cycles and frequency space which lies in the way in which economic fluctuations can be analysed and understood by decomposing the data into components of different frequencies. This approach, based on spectral analysis, makes it possible to identify variations in the time series of economic variables (such as employment, inflation, etc.) that follow cyclical or periodic patterns, typical characteristics of business cycles.

Keywords: spectral envelope, business cycle, frequency space, principal components, multivariate analysis, economy

1. Introduction

1.1 The spectral envelope

The advance of scientific knowledge in various fields has been decisive for the development and application of new quantitative techniques that, through the integration of computational elements, have contributed to a better understanding of problems and their solutions. As a result, a wide variety of multivariate statistical techniques have been developed for the analysis of the time series of continuous values over time and in the frequency domain.

In the time series analysis, there are two approaches to solving problems: using time space or frequency space. One or the other is used depending on what needs to be validated, estimated, or contrasted. For financial series, especially those closely related to the estimation of business cycles, frequency space has been used as an alternative. For this purpose, once the variables involved have been identified, it is desirable to control for variability and, therefore, it is possible to apply some kind of mathematical transformation to the set of selected series. There are several alternatives that can be used, but most of them are applied individually to each series. One suggested alternative is to use the spectral envelope from a multivariate perspective.

The spectral envelope, introduced by Stoffer, Tyller and McDougall [21], is one of the concepts used to analyse time series in the frequency domain. The primary objective of this statistical development was to enable the analysis of categorical time series, subsequently, it was implemented in multiple directions, thus facilitating its application to different areas [25].

1.2 The spectral envelope and data transform

Spectral enveloping is a transformation that analyses the frequency structure of the time series, while time series transformations are more general techniques that are applied to the data to make it more suitable for further analysis. Both provide a more detailed view of time series dynamics and are useful for both modelling and forecasting.

In other words, because they allow both time and frequency variations of signals to be understood and captured, spectral enveloping and time series transformations are fundamental tools for decomposing, analysing, and modelling complex time series.

For several fundamental reasons, it is important to note that time series transformations merit further investigation.

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Data complexity	Time series often exhibit complex behaviours, such as trends, seasonality, and cycles. Understanding how these features affect the dynamics of the data is critical to developing more accurate and efficient models
Improving predictive models	Appropriate transformations can significantly improve the predictive ability of models. Investigating different forms of transformation (logarithmic, differential, seasonal, etc.) can improve the accuracy of predictions.
Adapting to new technologies	With the advancement of machine learning and data analysis techniques, it is necessary to investigate new methodologies that integrate transformations to optimize the performance of these models on time series data.
Dealing with non-stationarity	Many time series are non-stationary, meaning that their statistical properties change over time. Research into transformations can help to find effective ways of stabilizing the variance and mean, making analysis easier.
Emerging challenges	New data-related challenges are constantly emerging in areas such as finance, economics, and the environment. Transformations can be useful tools to deal with issues such as volatility, abrupt changes, or complex correlations between variables.
Interdisciplinarity	Research on time series transformations can benefit from interdisciplinary approaches combining knowledge from statistics, mathematics, engineering, and social sciences, which can lead to new perspectives and innovative solutions.
Interpretation and communication	To communicate properly, it is essential to understand the data and transform them correctly.

Remember that the estimation of the spectral envelope depends on the transformations used, for which it is advisable to study the information and the way in which it has been obtained.

1.3 Data transformation and its importance in statistical work.

The most natural way to mathematically describe a cyclic motion is through the concept of Fourier series, and the formal representation of these series is given in a frequency space. In this complex space, some statistical techniques have been established to perform univariate and multivariate analysis, including principal component analysis, which is important in constructing the spectral envelope. To understand the theoretical and practical developments in this field, it is essential to review the relevant literature on time series data transformations.

Below is a selection of relevant topics and authors that can serve as a starting point for your research:

Fundamentals of time series	Comments
Time Series Analysis: Forecasting and Control, by George E. P. Box, Gwilym M. Jenkins and Gregory C. Reinsel [3].	This classic book lays the foundations of time series analysis and includes a chapter devoted to identifying the need for transformations to achieve stationarity.
Introduction to Time Series and Forecasting, by Peter J. Brockwell and Richard A. Davis [5].	Provides a clear approach to time series methods and discusses the importance of transformations.
Common transformations	
Log-log transformations.	These are commonly used to stabilize the variance of data that exhibits heteroscedasticity. The relevant literature includes papers on this transformation and its application in econometrics.
Differencing.	Fundamental in dealing with non-stationarity. Authors such as Hyndman and Athanasopoulos discuss differencing as a key technique in their book Forecasting: principles and practice.
Advanced models	
Applied Econometric Time Series, by Walter Enders [6].	Discusses autoregressive and moving average models, as well as the transformations needed to properly apply these models.
Elements of Statistical Learning, by Trevor Hastie, Robert Tibshirani and Jerome Friedman [18].	Although more focused on machine learning, discusses how transforming data can improve predictive models.
Seasonality and trends	
The decomposition of time series into seasonal, trend and noise components are an important topic.	Seasonal adjustment by x-13arima-seats in R.
Modern Methods	
State Space Models with Applications in Economics and Finance, by Andrew C. Harvey [9].	This book offers a modern approach to time series modelling, including transformations.
Recent research	
Look for articles in academic journals such as Journal of Time Series Analysis, Journal of Forecasting and International Journal of Forecasting, where up-to-date research on new transformations and methodologies in time series	

are published.	
Machine learning techniques	
The intersection between machine learning and time series analysis is receiving attention, authors such as Rob Hyndman and George Athanasopoulos [11].	Have addressed modern techniques that combine data transformations with machine learning algorithms in their work.
Case studies and applications	
Books such as Practical Time Series Analysis by Aileen Nielsen [15].	Provide practical examples of how transformations affect results in real-world situations.

To provide a solid foundation for understanding time series data transformations and to guide you in further exploration and analysis of the topic, we have created this literature reference. I invite you to search for specific articles and recent reviews to keep abreast of current trends in this evolving field.

1.4 Transformations and their relationship to the business cycle

Data transformations are essential tools for the analysis and estimation of business cycles, as they allow us to deal with problems such as non-stationarity, heteroskedasticity and stationarity in time series. The transformations performed can reveal hidden patterns in economic cycles (such as expansions and recessions) and these patterns can be used to inform policy and business decisions.

The use of data transformations allows problems inherent in time series data to be addressed, improving model specification and the quality of forecasts. However, it is crucial to bear in mind that each data series may require different types of transformations and models, so careful analysis tailored to the specific characteristics of the data is required.

2. Method

The concept of the spectral envelope is of a mathematical nature and was introduced by Stoffer et al [21, 16 pp. 224-253] in the analysis of time series and their periodicity for the spectral analysis and the scaling of categorical processes. Some years later, McDougall, Stoffer and Tyler [21] developed the spectral envelope concept for real time series. For the univariate case, let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ stationary time series in $\mathbb{R}$, and consider a time series transformation denoted by $g(X_t)$. In the case of a single variable, the main idea is to find transformations of some class $\mathcal{g}$ such that the resulting spectral density gives more information, in some quantifiable way, about the set of series under consideration. Let's consider the spectral density $f_g(w)$ of $g(X_t)$ at a frequency w and $\sigma_g^2 = \int f_g(w)dw$ is the total power of $g(X_t)$, It is noted that the transform of the time series that best highlights the frequency w is the one in which the

resulting standardized spectral density at w is relatively large, i.e., $f_g(w)/\sigma_g^2$, without losing generality, can be selected at the frequency $-\pi < w \leq \pi$.
Therefore, if we consider $\boldsymbol{X}_t$ with $t = 0, \pm 1, \pm 2, \dots$ a time series on $\mathbb{R}^p$ to $\mathbb{R}$ such that for any $g \in \mathcal{G}$ the spectral density $f_g(w)$ of $g(\boldsymbol{X}_t)$ exist, then the spectral envelope of $\boldsymbol{X}_t$, with respect to $\mathcal{G}$, is defined as

$$\lambda(w) = \sup_{g \in \mathcal{G}} \{f_g(w)/\sigma_g^2\}, -\pi < w \leq \pi \qquad (1)$$

For each transformation within class $\mathcal{G}$, the quantity $\lambda(w)dw$ represents the largest fraction of the total power that can be attributed to the frequencies wdw. It should be noted that the existence of $\lambda(w)$ only depends on whether or not the transformed time series have spectral densities, and since we generally have to take the supremum over a class of functions for each frequency, direct use of the definition to construct a spectral envelope is often complicated. Nevertheless, if $\mathcal{G}$ is a finite-dimensional vector space, then the problem is solved as a finite-dimensional eigenvalue problem for each frequency, in the case of a real-valued series, it is not immediately evident which basis of functions for $\mathcal{G}$ would be most appropriate. It shall be assumed that $\mathcal{G}$ is a k-dimensional vector space. It follows that there exists a set of basis functions $\{g_1, g_2, \dots, g_k\}$ so that for every $g \in \mathcal{G}$ so that

$$X_t(\beta) = \beta^\tau Y_t \qquad (2)$$

where $Y_t = \left(g_1(X_t), g_2(X_t), \dots, g_k(X_t)\right)^\tau$ is the vector process and $\beta = (\beta_1, \beta_2, \dots, \beta_k)^\tau \in \mathbb{R}^k$.

If the basis vector Y_t is assumed to possess a continuous spectral density, $f(\omega)$ say, then $X_t(\beta)$ will have a continuous spectral density $f(w;\beta)$ for all $\beta \in \mathbb{R}^k$. Moreover, (2) implies that $f_g(w) \equiv f(w;\beta) = \beta^\tau f(w)\beta$ and $\sigma_g^2 \equiv Var[X_t(\beta)] = \beta^\tau V\beta$ where $V = Var(Y_t) = \int_{-\pi}^{\pi} f(\omega)d\omega$ is the total power of Y_t. Therefore, the spectral envelope over $\mathcal{G}$ is well-defined and can be expressed as

$$\lambda(w) = \sup_{\beta \in \mathrm{B}} \{\beta^\tau f(w)\beta / \beta^\tau V\beta\}, -\pi < w \leq \pi \qquad (3)$$

where $\mathrm{B} = \{\beta / V\beta \neq 0\}$. If we consider only the values of the real numbers and the fact that lambda is Hermitian

$$\lambda(w) = \sup_{\beta \in \mathrm{B}} \{\beta^\tau f^{re}(w)\beta / \beta^\tau V\beta\}, 0 \leq w \leq \pi \qquad (4)$$

where $f^{re}(w)$ denote the real part of $f(w)$, and $f^{re}(-w) = f^{re}(w)$.
Thus, the relationship between $\lambda(w)$ and the optimal linear coefficients $\beta(w)$ can also be defined in terms of the largest scalar $\lambda(w)$, such that for some $\beta(w)$:

$$f^{re}(w)\beta(w) = \lambda(w)V\beta(w) \qquad (5)$$

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with $V\beta(w) \neq 0$. Note that if V has full rank, then the supremum is over all $\beta \neq 0$. It follows that $\lambda(w)$ is the largest eigenvalue associated with the deterministic equation.

$$|f^{re}(w) - \lambda V| = 0 \tag{6}$$

This allows the dimension of g to be reduced without losing information. In cases when $\boldsymbol{X}_t$ is a multivariate time series in Tiao et al. [24, pp. 11-37], and Brillinger [4, chap. 9] different types of solutions are reviewed and analysed like analysis of the principal components, and one of the most interesting classes of transforms are linear combinations of $\boldsymbol{X}_t$, see Tiao et al. [1] and in spectral analysis, this class forms a k-dimensional vector space of transformations. The most natural choice of basis functions is the k-components, i.e., $\boldsymbol{Y}_t = \boldsymbol{X}_t$ in (2), which is the subject of a discussion in Brillinger [2]. When a multivariate time series is normalized in such a way that the total power is the identity matrix, the following applies, Brillinger's first principal component spectrum can be interpreted as the spectral envelope, which also corresponds to the best rank one filtering of the multivariate time series. If we consider the above-mentioned considerations regarding (1), given a series on $\mathbb{R}^p$, stationary second-order p-vector with mean μ_X, summable auto covariance function $c_{XX}(u), u = 0, \mp 1, \ldots$ and the spectrum density matrix.

$$f_{XX}(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} c_{XX}(u) exp\{-i\lambda u\}, -\infty < \lambda < \infty \tag{7}$$

Thus, if we consider the estimation of the principal components in the space of frequencies, one has the jth eigenvector $v_j(\lambda)$ and $\rho_j(\lambda)$ the corresponding eigenvalue of $f_{XX}(\lambda)$ with $j = 1,2,\ldots,p$. So, to define the principal component series of **X**(t), using Shumway and Stofer [10] let $S_j(t)$ be a complex-valued univariate process of **X**(t) so that by using the representation of Cramer[3]

$$\mathbf{X}(\mathrm{t}) = \int e^{i\lambda t} d\boldsymbol{Z}_X(\lambda) \tag{8}$$

Then the series

$$\mathbf{S}(\mathrm{t}) = \sum \mathbf{b}(\mathrm{t}-\mathrm{u})\mathbf{X}(\mathrm{u}) \tag{9}$$

Where $\{\mathbf{b}(\mathrm{u})\}$ is a qxp filter, it follows that

$$\mathbf{S}(\mathrm{t}) = \int \boldsymbol{B}(\lambda) e^{i\lambda t} d\boldsymbol{Z}_X(\lambda) \tag{10}$$

With

[1] Op. cit.
[2] Op. cit.
[3] Op. cit.

Vol. 9, No.04; 2025

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$$\boldsymbol{B}(t) = \begin{bmatrix} \overline{v_1(\lambda)^T} \\ \cdot \\ \cdot \\ \cdot \\ \overline{v_1(\lambda)^T} \end{bmatrix}$$

Therefore, the j-th principal component series is

$$S_j(t) = \int \overline{v_j(\lambda)^T}\, e^{i\lambda t} d\boldsymbol{Z}_X(\lambda) \qquad (11)$$

Also $S_j(t)$ and $S_k(t), j \neq k,$ have zero coherency for all frequencies and if suppose that $\mathbf{X}(\mathrm{t}), t = 0,1, \ldots, T-1$, the estimate of $f_{\boldsymbol{XX}}(\lambda)$ has a form of

$$f_{\boldsymbol{XX}}^T(\lambda) = 2\pi T^{-1} \sum_{s=1}^{T-1} W^{(T)}\left(\lambda - \frac{2\pi s}{T}\right) \boldsymbol{I}_{\boldsymbol{XX}}^T\left(\frac{2\pi s}{T}\right) \qquad (12)$$

where $\boldsymbol{I}_{\boldsymbol{XX}}^T$ is the matrix of second periodograms

$$\boldsymbol{I}_{\boldsymbol{XX}}^T(\alpha) = (2\pi T)^{-1}\left[\sum_{t=0}^{T-1} \boldsymbol{X}(t)\, e^{-i\alpha t}\right]\left[\sum_{t=0}^{T-1} \boldsymbol{X}(t)\, e^{-i\alpha t}\right]^T \qquad (13)$$

And $W^{(T)}(\alpha)$ is a weight function defined by

$$W^{(T)}(\alpha) = \sum_{j=-\infty}^{\infty} W(B_T^{-1}[\alpha + 2\pi j]) \qquad (14)$$

With $W(\alpha)$ being concentrated in the neighbourhood of $\alpha = 0$ and $B_T, T = 1,2, \ldots$
This theoretical information allows us to find the main components, the basic ideas and theory of principal components analysis in frequency space can be reviewed in detail in Brillinger[4].

2.1 Advantages and disadvantages of the spectral envelope

The spectral envelope is a powerful analytical tool that enhances the understanding of complex time series and signal structures across multiple domains. One of its key advantages is its ability to identify cyclical patterns by decomposing a signal into its spectral components, revealing repetitive or seasonal behaviour that may not be apparent using traditional methods. This feature is particularly valuable in economics, meteorology, and systems engineering, where decision making often depends on identifying cycles in behaviour. In addition, the spectral envelope enhances frequency analysis by highlighting dominant frequency components while isolating

[4] Op. Cit.

less significant ones, providing a clearer representation of the fundamental structure of a signal. Another key benefit is its noise filtering capability, which improves data quality by separating relevant frequencies from extraneous noise, leading to more accurate interpretations. This is particularly important in fields where signal integrity is paramount, such as telecommunications and biomedical engineering. In addition, the spectral envelope supports trend prediction by modelling cyclic and frequency components, enabling anticipation of future data movements, and optimizing strategic planning in dynamic environments. Finally, it contributes to a more accurate interpretation of volatility by analysing fluctuations within a signal, providing valuable insights for risk assessment in financial markets and other areas dealing with uncertainty. In summary, the Spectral Envelope increases the depth and accuracy of data and signal analysis, making it a critical tool for improving decision making and ensuring more reliable interpretations of complex data sets, see [7],[23],[16],[8],[2].

2.2 Limitations and challenges of spectral envelope analysis in data processing

Despite its advantages, the spectral envelope presents several challenges that must be considered for its effective application in data and signal analysis. A major drawback is its complexity in interpretation, as spectral decomposition generates a large amount of mathematical and technical data that can be difficult to analyse, particularly for those without specialist training. This complexity not only increases the learning curve but can also delay decision making. In addition, the spectral envelope is highly sensitive to temporal resolution; an inappropriate choice of this parameter can either obscure critical details or introduce excessive noise, leading to inaccurate conclusions. Another limitation is its reliance on large data sets - when applied to small or temporally limited signals, results may lack reliability, reducing the effectiveness of the method in data-poor environments. In addition, the spectral envelope struggles with non-stationary time series, where statistical properties change over time. Since spectral analysis typically assumes stationarity, its application to dynamic time series in fields such as economics or meteorology can produce misleading or inconsistent results. Another critical issue is the risk of overfitting, where excessive reliance on spectral enveloping can lead to the capture of noise rather than meaningful trends, reducing the generalizability and predictive power of the model. Finally, the computational cost associated with spectral envelope calculations can be significant, particularly when dealing with large data sets or high-resolution signals. The high processing power required can be a challenge for real-time applications or environments with limited computing resources. Given these limitations, careful consideration must be given before implementing spectral envelope analysis, particularly in environments where data availability, computing capacity or expertise is limited, see [17],[20],[28],[12],[26],[14].

2.3 Cyclical Indicators System

From an economic perspective, the evolution of a country's economic activity over a given period of time is known as the business cycle. Its identification and analysis have been the subject of several studies using different methodologies [19]. These studies have shown that changes in economic activity can manifest themselves in two ways: through a period of relatively rapid growth and expansion, or through a period of decline and contraction. However, what is most relevant from a frequency domain analytical approach is that recent research using spectral analysis methods has confirmed the existence of business cycles in global Gross Domestic Product (GDP) [22]. Nevertheless, it is important to note that the main reference in this field remains the National Bureau of Economic Research (NBER), one of the leading institutions in economic research and the production of official statistics, which maintains a chronology of business cycles in the United States, identifying them based on absolute declines or changes in the aggregate level of output.

“The chronology identifies the months of peaks and troughs of economic activity. Expansions are the periods between a trough and a peak; recessions are the periods between a peak and a trough. By convention, the NBER classifies the peak month as the last month of expansion and the trough month as the last month of recession. Expansion is the normal state of the economy; most recessions are brief. However, the time that it takes for the economy to return to its previous peak level of activity, or its previous trend path may be quite extended.”

This is a methodology which is in use by many developing countries and their statistical offices. They produce official or government information on which political and social decisions are based.

However, there are alternatives, such as Baxter and King [1], who developed the business cycle approach based on spectral analysis of economic time series, which directly justifies spectral envelope estimation as a good alternative. As the benchmark of macroeconomic theory and policy is clearly the measurement of the business cycle, it is essential that we can define and identify the business cycle from multivariate information and generate benchmark cycles automatically [30]. In Mexico, the National Institute of Statistics and Geography (INEGI) presents the results of the Cyclical Indicators System (CIS) [29], see figure 1, which provides timely monitoring of the behaviour of the Mexican economy and offers information for the analysis of economic cycles. To illustrate the importance and estimation of the spectral envelope, we have selected information from this system, since it is a multivariate design.

Growth Cycle Approach: Coincident and Leading Indicators
January 1980 to October and November 2024
(points)

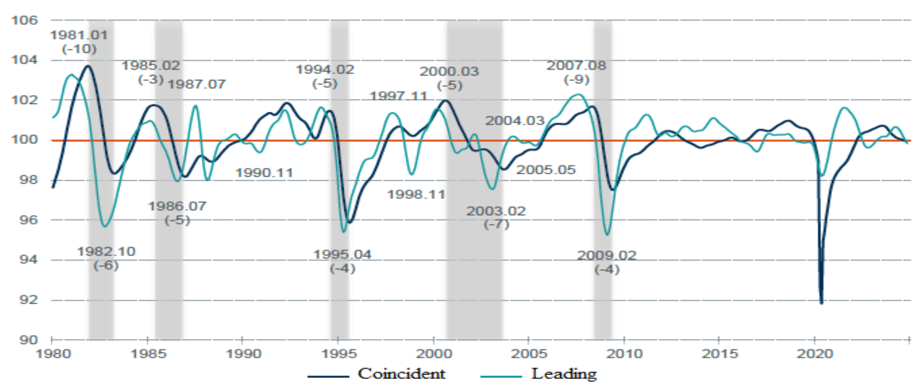

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Figure 1: INEGI. Cyclical Indicators System (CIS), 2024.[5]
Note: The coincident and leading indicators of the Mexican cyclical system, estimated by INEGI. The full period covered is 1980-2025.

The leading indicator has been constructed using the trend in manufacturing employment, the business confidence indicator: right time to invest, the price and price index of the Mexican Stock Exchange in real terms, the real bilateral Mexico-US exchange rate, the equilibrium interbank interest rate and the Standard & Poor's 500 index (US stock market index), see figure 2. The elements on which the leading indicator is based can be found in the same INEGI document referred to above. Without any loss of generality, as the indicators are updated over time, complete information is available for each indicator for the period 01-2004 to 11-2024 in the Economic Information Bank [19] (see BIE, INEGI), and from this information, work was done to estimate the spectral envelope, see figure 3.

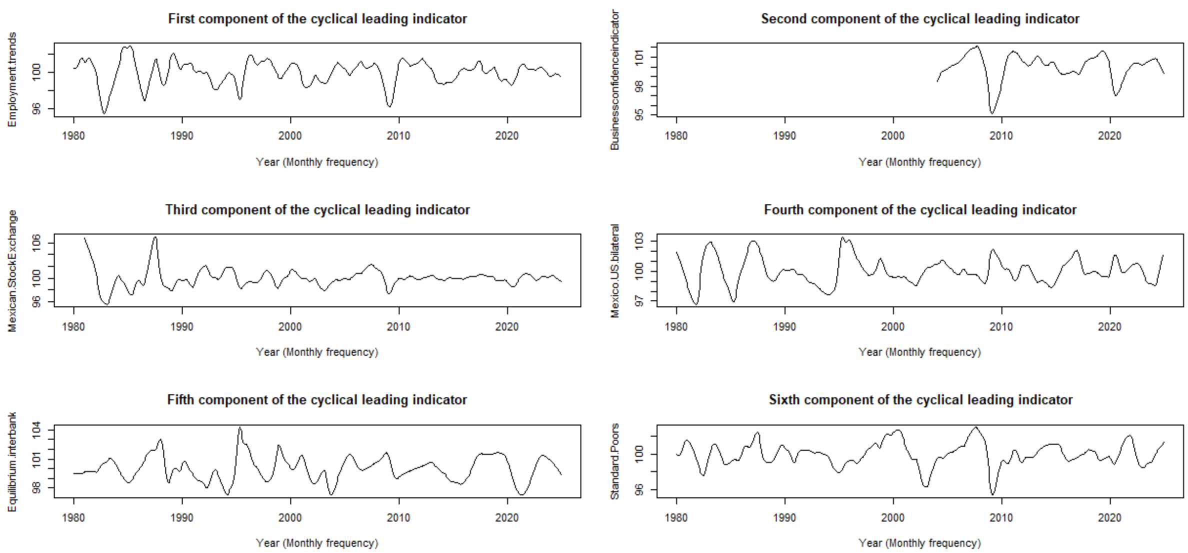

Figure 2: Representation of the components of the CIS, based on information provided by the BIE.

[5] *Notes*: The long-term trend of the coincident and leading indicators is represented by the line at 100. The numbers on the graph (e.g., 1981.01) represent the year and month in which the turning point occurred in the leading indicator: peak or trough.

The numbers in parentheses indicate the months that a given turning point of the leading indicator precedes the turning point of the coincident indicator. These numbers may change over time. The shaded area indicates the period between a peak and a trough in the coincident indicator. Series produced by econometric methods.

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Note: The dynamics over time of the six leading indicators of the cyclical system, with information available for each of the indicators selected.

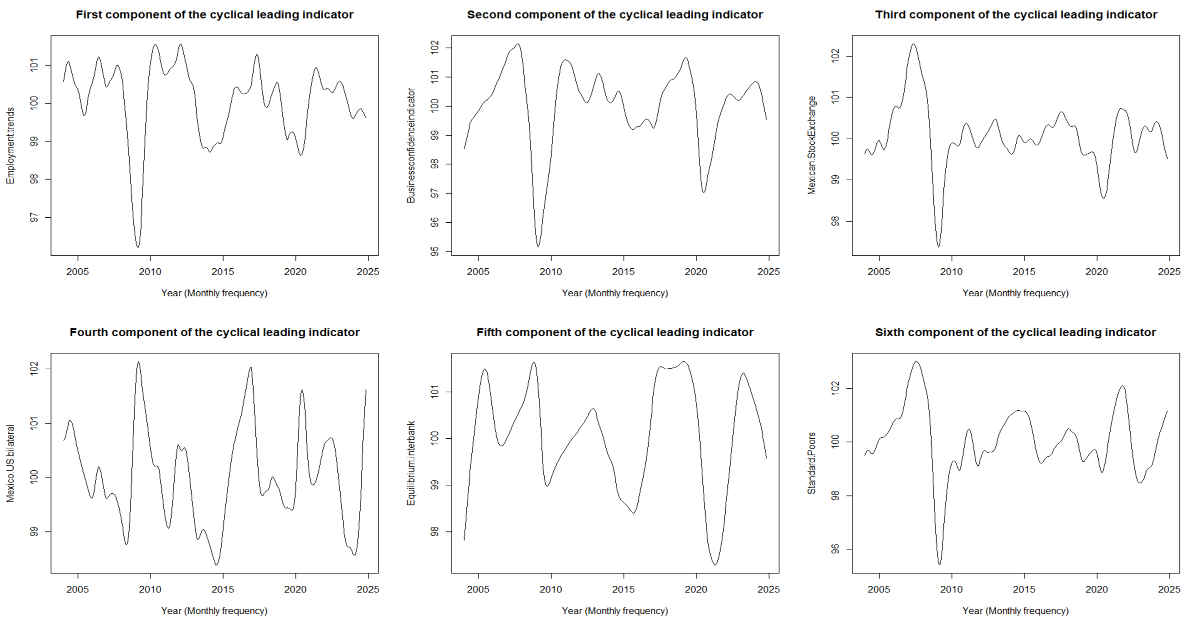

Figure 3: Representation of the components of the CIS, based on information provided by the BIE for the period 01-2004 to 11-2024.

Note: Temporal dynamics of the six indicators of the leading indicator of the cyclical system, with selected information, complete for all the years 01/2004 - 11/2024 for each selected indicator.
Each of the indicators used to estimate the forward component has been properly selected and constructed using econometric methods, and it can be observed that at least four of the six indicators have very similar dynamic behaviour. The construction of the leading indicator is analysed by cross-validation, so there should not necessarily be cointegration assumptions, and the corresponding ARIMA and the models in each series may or may not have a certain degree of similarity in the parameters [31], see figure 3. For more details on the construction of the leading indicator, please visit INEGI's web page on the system of cyclical indicators, where you will find a more detailed explanation of the series that make up the indicator and the main assumptions considered in its construction.

3. Results

A Fourier Transform is a mathematical operation that transforms a time-domain signal into a frequency-domain signal. In the time domain the signal is expressed in terms of time, and in the frequency domain it is expressed in terms of frequency. For the indicators chosen for the estimation of the spectral envelope, we found the spectrum in figure 4.

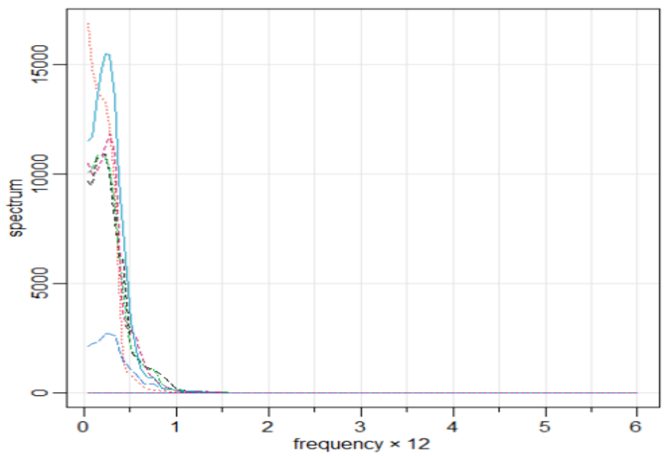

Figure 4. Spectrum of indicators forming the cyclical leading indicator system.

Note: The spectrum dynamics for indicators related to the cyclical leading indicator are shown, we know that, depending on the values of each indicator chosen, each spectrum will vary.

As an example of spectral envelope estimation, we work with the Yeo-Johnson transform [27].

$$\psi(y,\lambda) = \begin{cases} \frac{(y+1)^{\lambda}-1}{\lambda}, y \geq 0 \text{ } and \text{ } \lambda \neq 0 \\ \log(y+1), y \geq 0 \text{ } and \text{ } \lambda = 0 \\ -\frac{(-y+1)^{2-\lambda}-1}{2-\lambda}, y < 0 \text{ } and \text{ } \lambda \neq 2 \\ -\log(-y+1), y < 0 \text{ } and \text{ } \lambda = 2 \end{cases} \quad (15)$$

The Yeo-Johnson Transformation can be thought of as generalizing the Box-Cox Transformation [11, pp. 64].

Selecting $g_i \in g$ where $\{g_1 = \psi(y,1), g_2 = \psi(y,2), g_3 = \psi(y,-1), y \geq 0\}$

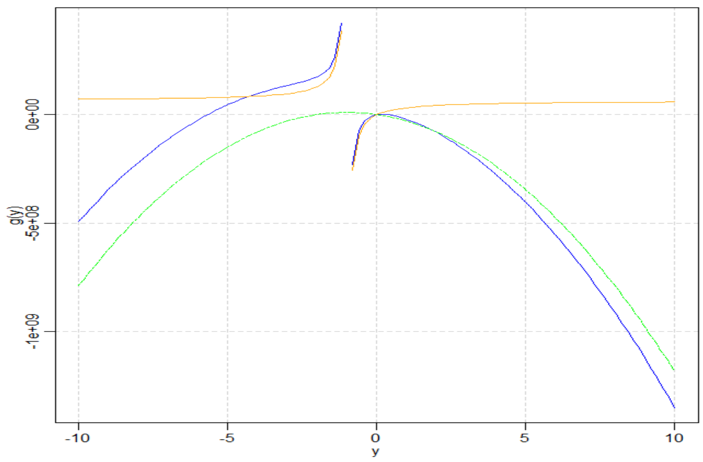

Figure 5. Optimum spectral.

Note: Optimum spectral envelope (blue) of indicators forming the cyclical leading indicator system.

In physics and engineering, the envelope of an oscillatory signal is a smooth curve that defines its extrema and provides a generalized representation of amplitude variations over time, space, angle, or other relevant variables, see figure 5. This concept is fundamental to signal processing as it allows the underlying patterns and frequency components within complex signals to be more clearly visualized. When applied to spectral data, Principal Component Analysis (PCA) facilitates the extraction of the dominant spectral envelope, effectively capturing the most significant common frequency components across the data set. This process results in a smoothed envelope that improves the interpretability of the signal by highlighting its essential spectral features while reducing noise and less relevant fluctuations. The integration of spectral envelope analysis with PCA thus provides a powerful approach to dimensionality reduction and feature extraction, improving signal characterization in various scientific and engineering applications, including telecommunications, biomedical engineering, and structural health monitoring. By isolating key frequency components, this method contributes to more accurate modelling, classification and decision making in complex systems.

The estimation of the spectral envelope has been achieved by implementing a Daniell filter (2, 2) that provides the coefficients that would be used as weights in averaging the original data values for a convolved Daniell kernel with m = 2 in both smoothing. This method can be adapted according to the type of analysis carried out on the spectrum derived from the selected set of time series.

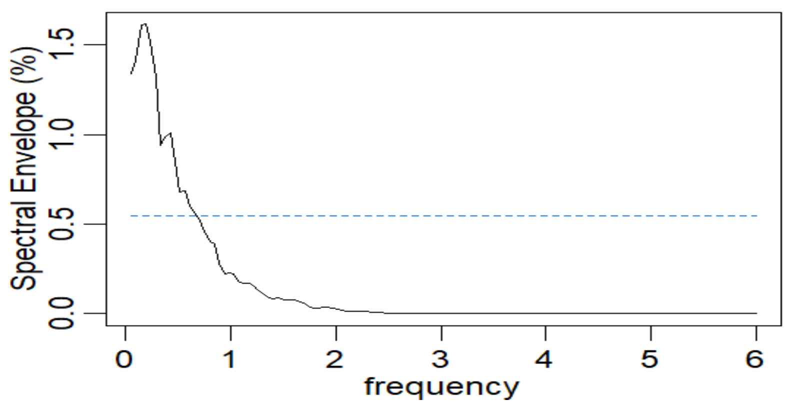

Figure 6. It is the percentage of the spectral envelope for the frequencies.

Note: Spectral envelope of the indicators that make up the system of leading cyclical indicators extracted using the PCA method.

In the forward cyclical component, 6 indicators are used, and three elements were selected to form the family of functions, i.e., 18 estimates of the beta vector were obtained.

Table 1. The estimation of the betas with respect to the g_i functions.

Beta Coefficients					
β_1	β_2	β_3	β_4	β_5	β_6
600.40	**-4924.71**	**-1815.02**	**5782.69**	**10807.67**	**-16991.29**
β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
-3.95	**32.70**	**11.81**	**-37.29**	**-71.78**	**111.95**
β_{13}	β_{14}	β_{15}	β_{16}	β_{17}	β_{18}
-2046623.02	**16545721.79**	**6342972.63**	**-20569031.73**	**-36282788.39**	**57976316.07**

Note. The estimates are based on $\psi(y,1), \psi(y,2), \psi(y,-1)$ respectively.

When considering the multiple simulation options available for non-deterministic phenomena, it is essential to recognize the inherent randomness in data due to complex, difficult-to-control factors that reflect human behaviour in relation to survival. This aspect, which is characterized by its dynamics and variability, necessitates a rigorous evaluation approach. A pertinent example of this phenomenon can be observed in economic cycles, highlighting the relevance of incorporating this complexity into statistical models. In this context, the construction of the spectral envelope, as an essential component in statistical estimation, is emphasized.

4. Discussion

Economic analysis is a complex subject, and one of the aspects on which decisions are based is the estimation and recognition of a country's economic cycles, since this is the origin of the concept of economic recession, and its prevention is usually fundamental for economic recovery. As we can see, there are different methods for estimating these cycles; the most appropriate ones are those that are mathematically and statistically robust and, above all, that ensure an appropriate treatment of the information used in the models. Therefore, since we are dealing with a multivariate phenomenon, it is more appropriate to use theoretical elements of the same style than to deal with an appropriate transformation for each indicator individually.

Therefore, in the application of transformations to the indicators involved in the estimation, the spectral envelope must become important in the modelling of business cycles. Obviously, the feasibility or otherwise of its application depends on each scenario, but this does not diminish its importance.

In many cases, the confidentiality of the data makes it difficult to obtain official statistical information, which was the case in the present study, but the aim was to illustrate how we can use the concept of the spectral envelope in a problem with real data.

Finally, it is an interesting project that proposes the work with many series that could be candidates in the construction of the economic cycles of a country, and that before the construction of its estimation allows the adjustment of the series under a scheme of a multivariate phenomenon, as is the construction of the economic cycles.

5. Conclusion

Spectral envelope analysis is a valuable tool for understanding economic dynamics, allowing the identification of trends and cycles in financial variables over time. However, its effectiveness is limited in highly volatile, non-stationary environments or with complex data structures, where complementary methods may be required to obtain a more accurate representation of the phenomena being analysed. Despite these limitations, the information provided by the spectral envelope provides an analytical framework for optimal economic decision making. Its use in the identification of short- and long-term economic cycles facilitates the formulation of adaptive strategies by businesses, investors and governments, allowing them to anticipate changes in economic activity and optimize their policies to maximize profits and minimize risks. Similarly, its integration into the design of monetary and fiscal policies contributes to macroeconomic stability by enabling better management of inflation, unemployment and economic growth. In this sense, the development of hybrid approaches that combine spectral envelopment with advanced data analysis techniques could improve its applicability in more challenging scenarios. This would increase its predictive capacity and enhance its usefulness in strategic planning and the formulation of more resilient economic policies.

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