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A STUDY ON ROBUST ESTIMATORS FOR GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTIC MODELS

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Abstract

Financial time series are often found to be heavy-tailed and skewed. Motivated from Skewed-t and Asymmetric Laplace distributions, two new M-estimators, called ST and AL, respectively, are introduced for estimation of GARCH-type models. Performance of estimators is checked with commonly used quasi-maximum likelihood, least absolute deviation and other robust estimators, for both symmetric and asymmetric models through a Monte Carlo study. Results of simulation revealed that both estimators provide accurate parameter estimates of GARCH models outperforming competing estimators when errors are generated from non-normal distributions. An application to real data set shows that these estimators also give better Value-at-Risk forecasts.

Keywords: M-estimators, Skewed-t, Asymmetric Laplace, GARCH, robust.

1 INTRODUCTION

Instantaneous variability or volatility is an important concept in financial time series. In aninfluential paper, the autoregressive conditional heteroscedastic (ARCH) model was introduced by Engle (1982)in which the volatility of current asset return is represented as the linear function of past squared returns. This model could capture some empirical stylised facts such as heavy-tailedness, volatility clustering and time varying variability which are associated with financial time series. Many other extensions have been proposed since the introduction of the ARCH model. Among those the generalized ARCH (GARCH) model of Bollerslev (1986) is the most popular and widely-used.

The impact on conditional variance, as a result of variations of unexpected returns, plays a significant role in modeling volatility. However, in the GARCH model it is observed that the unexpected rise gives limited contribution towards the conditional variance than the unanticipated decrease. Glosten et al. (1993) introduced an asymmetric GARCH model also known as the GJR model. Consider the returns$y_t;\ 1 \le t \le n$, such that

$$y_t = \sigma_t \varepsilon_t, \qquad (1)$$

where $\{\varepsilon_t: t \in Z\}$ are unobservable identical and independently distributed errors which are symmetric about zero. In the GJR(1, 1) model, the conditional volatility is defined as

$$\sigma_t{}^2 = \beta_0 + \beta_1 y_{t-1}^2 + \beta_2 \sigma_{t-1}^2 + \beta_3 D_{t-1} y_{t-1}^2, \qquad (2)$$

where $D_{t-1} = I(y_{t-1} < 0)$ with $I(\cdot)$ as an indication function and $\theta_o = (\beta_0, \beta_1, \beta_2, \beta_3)'$ is the vector of unknown parameters in a parameter space, such that

$$\Theta = \{\theta = (\beta_0, \beta_1, \beta_2, \beta_3)'; \beta_0, \beta_1, \beta_2 > 0, \beta_1 + \beta_3 \geq 0, \beta_1 + \beta_2 + \beta_3/2 < 1\}.$$

Under these conditions, the model stated in equation (1) and (2) is known to be strict stationary.

In the GJR(1, 1) model the positive returns increase the volatility through factor β_1 while negative returns contribute through factor $\beta_1 + \beta_3$. Henceβ_3 is said to be the skewness parameter and $\beta_3 > 0$displays the leverage effect. The GJR (1, 1) model with$\beta_3 = 0$,reduces to the GARCH(1, 1) model. Asymmetric models other than the GJR model include the threshold GARCH (TGARCH)model by Zakoian(1990), Exponential GARCH (EGARCH) model given by Nelson(1991)and asymmetric GARCH (AGARCH) model of Engle and Ng(1993). The GARCH (1, 1) model was also introduced as Autoregressive Moving Average Conditional Heteroscedastic (ARMACH) model in 1986 by Taylor (2008).

A well-known approach for the estimation of unknown parameters of GARCH-type models is by using Gaussian likelihood for the innovation $\{\varepsilon_t\}$ and the estimator obtained is known as Quasi Maximum Likelihood Estimator(QMLE). This estimator is asymptotically normal and consistent assuming that the innovation contains finite four moments. However, in many situations such strict conditions are not supported like Student-*t* distribution in which degrees of freedom is at most four. Many authors introduced various robust estimators for the estimation of the unknown parameters of GARCH model. Robust estimators are based on less strict conditions of moments and not influenced by the small fraction of outliers.

Peng and Yao (2003) proposed an estimator based on absolute deviations through logarithmic transformations and termed as the Least Absolute Deviation (LAD) estimator. Berkes and Horvath (2004) introduced a class of robust estimators for GARCH model. Muler and Yohai (2008) developed two types of robust estimates namely M-estimator and bounded M-estimator. Mukherjee (2008) studied the asymptotic normality of the class of robust M-estimators of GARCH model. Iqbal and Mukherjee (2010) extended the M-estimation approach to other GARCH-type models using various estimators such as the QMLE, LAD, Cauchy, Huber and B-estimators. Carnero et al. (2012) studied the effect of extreme values on the GARCH model volatilities by analysis of maximum likelihood, QMLE, bounded M-estimator and bounded QMLE. Huang et al. (2015) introduced M-estimators using GJR scaling model by considering intraday high frequency data. Iqbal (2017) used M-estimators for the evaluation of volatility and Value-at-Risk (VaR) forecasting of Karachi Stock Exchange (KSE) during and after the global financial crisis.

In this article, we introduce two new M-estimators as alternatives to QMLE, LAD and B-estimators for the estimation of the unknown parameters of GARCH-type models. These new estimators, termed ST and AL estimators, motivated from the Skewed-*t* and Asymmetric Laplace densities, respectively, may be helpful in dealing with heavy-tailedness, extreme observations and asymmetry commonly found in financial data. The performance of newly proposed estimators is compared and evaluated with other robust estimators through a Monte Carlo study. Our results revealed that both ST and AL estimators provide accurate estimates than all other candidate estimators under study for the parameters of GARCH-type models when errors are heavy-tailed and skewed. A real data application is also provided where the VaR forecasts for the Karachi Stock Exchange data are obtained. Various evaluation criteria have been used for the evaluation of in-sample and out-of-sample VaR estimates and results of empirical application revealed that both A Land ST estimators provided reliable risk estimates and outperformed competing estimators.

The rest of the paper is organized as follows: In Section 2, the robust M-estimators of GARCH-type models are defined. Section 3 discusses estimation and prediction of VaR along with various back testing approaches of VaR evaluation. The results of simulation study and application to real data set are presented in Section 4. Finally, Section 5 concludes the article.

2 ROBUST M-ESTIMATORS

By recursive substitutions, from (2), we get

$$\sigma_t^2 = \frac{\beta_0}{(1-\beta_2)} + \beta_1 \sum_{j=1}^{\infty} \beta_2^{j-1} y_{t-j}^2 + \beta_3 \sum_{j=1}^{\infty} \beta_2^{j-1} D_{t-j} y_{t-j.}^2$$

For $\theta \epsilon \Theta,$ the variance function is defined as

$$v_t(\theta) = \frac{\beta_0}{(1-\beta_2)} + \beta_1 \sum_{j=1}^{\infty} \beta_2^{j-1} y_{t-j}^2 + \beta_3 \sum_{j=1}^{\infty} \beta_2^{j-1} D_{t-j} y_{t-j.}^2 \quad (3)$$

and $\sigma_t = v_t{}^{1/2}(\theta_o)$. In location model, the M-estimators are stated to be the solution to the equations of mean residual functions. Therefore, in scale estimation problem, the M-estimator is regarded as the solution of the equations of residual variance function$v_t(\theta)$. If the error density is denoted by f in(1), then the conditional density for $y_t$given available information till $t-1$is $v_t{}^{-\frac{1}{2}}(\theta_o) f\left(v_t{}^{-1/2}(\theta_o) y_t\right)for1 \le t \le n$. Therefore, by maximum likelihood estimation a random quantity is defined which minimizes the negative log-likelihood function $\frac{1}{n}\sum_{t=1}^{n}\left[(1/2)\log v_t(\theta) - \log f\left(y_t / v_t{}^{1/2}(\theta_o)\right)\right], \theta \epsilon \Theta$ or just as gradient function solution:

$$\sum_{t=1}^{n} \left(\frac{1}{2}\right)\left[1 - H^*\left(\frac{y_t}{v_t^{\frac{1}{2}}(\theta_o)}\right)\right]\left(\dot{v}_t(\theta)/v_t(\theta)\right) = 0,$$

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where:$H^*(y) := y\left(-\dot{f}(y)/f(y)\right)$and for a function f,$\dot{f}$ denotes the derivative or the gradient. Huber (1964) proposed M-estimator in which the equation of likelihood consisted the derivative of log-likelihood function in location models. A score function $H$was substituted in place of $-\dot{f}/f$ and the solutions of the equations are defined to be the M-estimators. Here, the estimation of parameters for scale is considered and the M-estimators are defined using score function H. Suppose$\psi: R \to R$ denote a function under skew symmetric structure i.e. $\psi(-y) = -\psi(y) \forall y \in R - \{0\}$,i.e. for finite number of points differentiable. Let$H(y) = y\psi(y), y \in \mathbb{R}$ such that $H(-y) = H(y), \forall$ □. We may define$\hat{\theta}_n$the estimate of the true parameter $\theta_n$of (1) and(2) as the solution of the following:

$$\sum_{t=1}^{n}\left(\frac{1}{2}\right)\left[1 - \mathrm{H}\left(\frac{y_t}{v_t^{\frac{1}{2}}(\theta_o)}\right)\right]\left(\dot{v}_t(\theta)/v_t(\theta)\right) = 0.$$

The observable approximations $\hat{v}_t(\theta)$ for $t \geq 1$ is defined as

$$\hat{v}_t(\theta) = \frac{\beta_0}{1-\beta_2} + I(t \geq 2)\left(\beta_0 \sum_{j=1}^{t-1} \beta_2^{j-1} y_{t-j}^2 + \beta_3 \sum_{j=1}^{t-1} D_{t-j}\beta_2^{j-1} y_{t-j}^2\right).$$

The M-estimator $\hat{\theta}_n$is then the solution of

$$\widehat{\mathrm{M}}_n(\theta) = \sum_{t=1}^{n} \widehat{\mathrm{m}}_t(\theta) = 0, \qquad (4)$$

Where $\widehat{\mathrm{m}}_n(\theta) = (1/2)[1 - H\{y_t/\hat{v}_t^{\frac{1}{2}}(\theta)\}]\{\dot{\hat{v}}_t(\theta)/\hat{v}_t(\theta)\}$. It emerges that $\hat{\theta}_n$estimates the function of true parameter θ_0, defined to be

$$\theta_{0H} = (c_H\beta_0, c_H\beta_1, \beta_2, c_H\beta_3)', \qquad (5)$$

where the constant $c_H > 0$ depends on the score function under study.

Next, we define various M-estimators.

Case 1: *ST estimator:*The Skewed-*t*distribution with degrees of freedom is $2 < v < \infty$and skeweness parameter $-1 < \lambda < 1$, is given by

$$f(y|v,\lambda) = bc\left[\left(1+\frac{1}{v-2}\left\{\frac{by+a}{1-\lambda}\right\}^2\right)^{-\frac{(v+1)}{2}} I\left(y < -{}^{a}/_{b}\right) + \left(1+\frac{1}{v-2}\left\{\frac{by+a}{1+\lambda}\right\}^2\right)^{-\frac{(v+1)}{2}} I\left(y \geq -{}^{a}/_{b}\right)\right].$$

Thus, for ST estimator $\psi(y) = \frac{(v+1)b}{v-2}\left[\left\{\frac{\left(\frac{by+a}{1-\lambda}\right)}{k_1}\right\} I\left(y < -{}^{a}/_{b}\right) + \left\{\frac{\left(\frac{by+a}{1+\lambda}\right)}{k_2}\right\} I\left(y \geq -{}^{a}/_{b}\right)\right]$ and $H(y) = yb\,\frac{(v+1)}{v-2}\left[\left\{\frac{\left(\frac{by+a}{1-\lambda}\right)}{k_1}\right\} I\left(y < -{}^{a}/_{b}\right) + \left\{\frac{\left(\frac{by+a}{1+\lambda}\right)}{k_2}\right\} I\left(y \geq -{}^{a}/_{b}\right)\right]$

where $k_1 = \left(1+\frac{1}{v-2}\left\{\frac{by+a}{1-\lambda}\right\}^2\right)$, $k_2 = \left(1+\frac{1}{v-2}\left\{\frac{by+a}{1+\lambda}\right\}^2\right)$,
$a = 4\,\lambda c\left(\frac{v-2}{v-1}\right)$, $b^2 = 1 + 3\lambda^2 - a^2$ and $c = \frac{\Gamma(v+1/2)}{\sqrt{\pi(v-2)}\Gamma(v/2)}$.

Case 2: *AL estimator:* The standard Asymmetric Laplace distribution with unit variance and zero mean is defined by $f(y|p) = b_p \exp\left[b_p|y|\left(\frac{1}{p}I(y<0) + \frac{1}{1-p}I(y>0)\right)\right]$ where p is usually close to 0.5 and denotes the shape parameter of the density. Thus, for Asymmetric Laplace estimator

$\psi(y) = b_p\frac{|y|}{y}\left(\frac{1}{p}I(y<0) + \frac{1}{1-p}I(y>0)\right)$ and $H(y) = b_p|y|\left(\frac{1}{p}I(y<0) + \frac{1}{1-p}I(y>0)\right)$
where $b_p = \sqrt{p^2(1-p^2)}$.

Case 3: *QMLEstimator:* $\psi(y) = y$, and the score function is $H(y) = y^2$.

Case 4: *LAD estimator:* $\psi(y) = \mathrm{sign}(y)$, and $H(y) = |y|$.

Case 5: *B-estimator:* $\psi(y) = B\,\mathrm{sign}(y)/(1+|y|)$, such that $B > 1$ is constant which is known, then $H(y) = B|y|/(1+|y|)$.

In this study, these M-estimators are employed for the estimation of GARCH-type models. The computation of these estimators is a crucial problem. We applied the algorithm defined in Iqbal & Mukherjee (2010) which is applicable for the computation of all score functions.

3 VALUE-AT-RISK (VAR)

Value-at-Risk is an approach of measuring the expected loss in a portfolio within a given interval of confidence upon the aimed horizon as an outcome of the adverse movements in relevant securityprices (Jorion 2000). Thus,for the known probability p, a $(1- \ p)100\%$VaR is stated to be the p^{th} conditional quantile of returns. Therefore, VaR at the time $t > 1$, for returns such that $y_t; 1 \le t \le n$is defined to be

$$q_t = \inf\{y: p \le P_{t-1}(y_t \le y)\}.$$

Here P_{t-1} denotes the conditional distribution of y_t, provided that the information is available till $t-1$. Therefore, from (1) we have $q_t = v_t(\theta_0)F^{-1}(p)$, the F^{-1} denote innovations $\{\varepsilon_t\}$ quantile function. Then the VaR estimate is stated to be

$$\widehat{q_t} = \hat{v}_t(\hat{\theta}_n)(np+1)^{th} \text{ order statistic of } y_t/\hat{v}_t\ (\hat{\theta}_n) \text{ for } 2 \le t \le n.$$

Value-at-Risk is the most widely used approach for estimation of the market risks. VaR enables the risk managers and the financial analyst to make forecast regarding the highest or worst portfolio's possible risk for the specified time-period at a given level of significance. Several evaluation procedures along with back testing methods are used for evaluation of VaR estimates.

3.1 Coverage Probability and Rate of Violation

Supposethe total violation observations are$n$. Let us define $n_* = \sum_{t=2}^{n} l_t$,where $l_t = I(y_t \le \widehat{q_t})$, then the probability of the empirical rejection is $p^{\wedge} = n_* \ / \ n$ to p is utilized for the analysis of overall predictive performance upon the model of VaR. This probability is also defined as the rate of violation of VaR supplying the VaR forecast insight.

Models in which the conditional quantile returns are correctly specified, true $\hat{p}$ will be equal to p. To develop a comparison of the competing models the ratio of $\hat{p}/p$ can be utilized. The model whose value is near to unity is preferred. If there are ties, then conservative model for $\hat{p}/p < 1$ is selected to be superior.

Average Quadratic Loss

In the evaluation of VaR, the magnitude of losses is also significant. This magnitude of losses has been studied by Lopez (1999) who explained the Average Quadratic Loss (AQL) of an estimate of VaR as $AQL = \sum_{t=1}^{n} QL_t/n$,where

$$QL_t = \begin{cases} 1 + (\widehat{q_t} - y_t)^2 & \text{if } y_t \le \widehat{q_t} \\ 0 & \text{if } y_t > \widehat{q_t} \end{cases}$$

3.2 Coverage Test

Kupiec (1995) stated the unconditional likelihood ratio test as

$$LR_{UC} = 2(\ln\{(1-\hat{p})^{n-n_*}\hat{p}^{n_*}\} - \ln\{(1-p)^{n-n_*}p^{n_*}\}),$$

where LR_{UC} is asymptotically $\chi^2_{(1)}$. Christoffersen (1998) introduced an independent coverage statistic which is denoted by LR_{IND}. For $i = 0, 1$ and $j = 0, 1$ suppose n_{ij} denote observations of time point such that $t; 2 \leq t \leq n$ among which $I_t = i$ is being followed by $I_{t+1} = j$. Now, Let$\hat{\pi}_{ij} = n_{ij}/((n_{i0} + n_{i1}))$, and $\hat{\pi} = ((n_{01} + n_{11}))/n$,then

$$LR_{IND} = 2\left(\ln\left((1-\hat{\pi}_{01})^{n_{00}}\hat{\pi}_{01}^{n_{01}}(1-\hat{\pi}_{11})^{n_{10}}\hat{\pi}_{11}^{n_{11}}\right) - \ln\left((1-\hat{\pi})^{(n_{00}+n_{10})}\hat{\pi}^{(n_{01}+n_{11})}\right)\right)$$

Christoffersen (1998) conditional coverage test statistic is asymptotically $\chi^2_{(2)}$and is defined as

$$LR_{CC} = LR_{UC} + LR_{IND}.$$

3.3 Dynamic Quantile Test

Engle and Manganelli (2004) introduced the dynamic quantile (DQ) test to investigate higher order dependence in VaR. The DQ test is defined as follows:

Let $h_t, 2 \leq t \leq n$,be the t^{th} response such that is

$$h_t = \begin{cases} -p & \text{if} y_t > \hat{q_t} \\ 1-p & \text{if} y_t \leq \hat{q_t} \end{cases}$$

and$h_1 = -p$. Now considering the linear regression model having the response $X = [h_1, h_2, \dots, h_n]'$ and a design matrix of order $n \times j$ where $Y = [y_{t,k}]$ with $j = 7$ and consisting the first column of ones. For$(t, k)^{th}$ term, if $k \geq t$, then $y_{t,7} = \hat{q}_t$, and if $k < t$, then $y_{t,k} = h_{t-k}$ where $2 \leq k \leq 6$.Then, the test DQ is asymptotically distributed as$\chi^2_{(7)}$ and is defined as

$$DQ = \frac{\hat{\beta}'Y'Y\hat{\beta}}{p(1-p)};$$

Here $\hat{\beta} = (Y'Y)^{-1}(Y'X)$ is the estimator of ordinary least square.

3.4 Mean Relative Bias

For thek^{th} estimator$(1 \leq k \leq c)$, the mean relative bias (MRB) defined by Hendrick (1996) is given by

$MRB_k = \frac{1}{n}\sum_{k=1}^{n}\frac{\hat{q}_{kt}-\overline{q_t}}{\overline{q_t}}$,where$\overline{q_t} = \frac{1}{c}\sum_{k=1}^{c}\hat{q}_{kt}$ and *c* denoted number of competing VaR estimates.

4 RESULTS AND DISCUSSION

4.1 Monte Carlo Simulation

The results of the Monte Carlo simulation are based on*R=1000*independent replications each of sample size *n=2000* from model (1) and (2) with the error generated from four different distributions. The distribution chosen for errors include the standard normal distribution, Student-*t* distribution with 4 degrees of freedom, contaminated normal distribution $(1-\epsilon)\square(x)+\epsilon\square(x/\sigma)$with $\epsilon = 0.05$ and $\sigma^2 = 9$and Skewed-*t* distribution with 4 degrees of freedom and skewness $\lambda = -0.25$. For the GARCH (1,1) model the true parameter values were $\beta_0 = 0.005, \beta_1$ =0.2 and $\beta_2 = 0.75$whereas for the GJR (1, 1) model the true parameter valuesare set as$\beta_0 = 0.5, \beta_1 = 0.3$, $\beta_2 = 0.4$ and $\beta_3 = 0.25$ as these are commonly observed empirical applications.

For the comparative analysis of all M-estimators based on different score functions, the mean squared errors (MSE), root mean squared errors (RMSE) and mean absolute errors (MAE) for the estimators for the GJR (1, 1) model is given by

$$\mathrm{MSE} = \mathrm{E}\left[\left\{\frac{(\hat{\beta}_0+\hat{\beta}_3)}{\hat{\beta}_1}+\hat{\beta}_2\right\}-\left\{\frac{(\beta_0+\beta_3)}{\beta_1}+\beta_2\right\}\right]^2$$

$$\mathrm{RMSE} = \sqrt{\mathrm{E}\left[\left\{\frac{(\hat{\beta}_0+\hat{\beta}_3)}{\hat{\beta}_1}+\hat{\beta}_2\right\}-\left\{\frac{(\beta_0+\beta_3)}{\beta_1}+\beta_2\right\}\right]^2}$$

$$\mathrm{MAE} = \mathrm{E}\left[\left|\left\{\frac{(\hat{\beta}_0+\hat{\beta}_3)}{\hat{\beta}_1}+\hat{\beta}_2\right\}-\left\{\frac{(\beta_0+\beta_3)}{\beta_1}+\beta_2\right\}\right|\right]$$

Similarly, these error functions can be defined for the GARCH (1, 1) model by setting $\beta_3 = 0$.

Table 1 and Table 2 show the estimated values of MSE, MAE and RMSE for the GARCH (1,1) and GJR (1,1) model, respectively, with standard errors presented in parenthesis based on 1000 replications. The bold entries in Table 1 and Table 2 represent least values for each row.

From Table 1, it is evident that for normally distributed errors the QMLE gives the best performance in terms of lowest MSE, MAE and RMSE. However, in case of heavy-tailed contaminated normal and skewed distributions, the results obtained are quite different. For the GARCH model, the AL estimator proves to be the best performing when errors were generated from Student-*t* distribution. For contaminated normal and Skewed-*t*errors, the ST estimator outperforms all candidate estimators in the analysis.

Table 1
MSE, MAE and RMSE of M-estimators of the GARCH (1, 1) model

	QMLE	LAD	B-Estimator	ST	AL
Normal Distribution					
MSE	**0.0036** (0.0054)	0.0045 (0.0073)	0.0074 (0.0163)	0.0069 (0.0176)	0.0055 (0.0131)
MAE	**0.0469** (0.0369)	0.0525 (0.0417)	0.0645 (0.0571)	0.0618 (0.0555)	0.0549 (0.0499)
RMSE	**0.0597**	0.0670	0.0861	0.0829	0.0742
Student-t Distribution					
MSE	0.0077 (0.0119)	0.0051 (0.0091)	0.0068 (0.0160)	0.0057 (0.0139)	**0.0049** (0.0093)
MAE	0.0703 (0.0524)	0.0560 (0.0442)	0.0613 (0.0553)	0.0566 (0.0499)	**0.0552** (0.0436)
RMSE	0.0877	0.0714	0.0826	0.0754	**0.0704**
Contaminated Normal Distribution					
MSE	2.5845 (5.8204)	0.0160 (0.0561)	0.0074 (0.0271)	**0.0044** (0.0129)	0.0116 (0.0387)
MAE	0.1906 (1.5971)	0.0845 (0.0943)	0.0599 (0.0621)	**0.0491** (0.0445)	0.0773 (0.0747)
RMSE	1.6076	0.1266	0.0863	**0.0663**	0.1075
Skewed-t Distribution					
MSE	0.0072 (0.0104)	0.0044 (0.0069)	0.0063 (0.0149)	0.0228 (0.5163)	**0.0042** (0.0092)
MAE	0.0670 (0.0518)	0.0518 (0.0415)	0.0591 (0.0531)	0.0614 (0.1379)	**0.0497** (0.0417)
RMSE	0.0847	0.0664	0.0795	0.1509	**0.0649**

Note: S.E. in the parenthesis. Bold values show minimum value for each row.

Similar results are observed from Table 2 where the QMLE is again found to produce accurate estimates for the parameters of the GJR (1, 1) model when errors are normally distributed. Again, for heavy-tailed, contaminated normal and skewed distributions, the QMLE does not perform well. For all these cases, the ST estimator is found to be the best choice. The analysis of Peng and Yao (2003) recommended that when the errors comprise of heavy tailed distributions the LAD estimator should be used. However, the study of Iqbal and Mukherjee (2010) indicated that B-estimator gives better performance than the LAD estimator in case of heavy tailed errors. The findings of this study reveal that there are estimators that can outperform both the LAD and B-estimators. The results show that, in case of non-normal errors, our proposed estimators ST and AL can be preferred over other robust estimators for the estimation of symmetric and asymmetric GARCH models.

Table 2
MSE, MAE and RMSE of M-estimators in the GJR $(1,1)$ model

	QMLE	LAD	B-estimator	ST	AL
			Normal Distribution		
MSE	**0.0021** (0.0018)	0.0033 (0.0088)	0.0043 (0.0126)	0.0027 (0.0136)	0.0034 (0.0139)
MAE	**0.0327** (0.0369)	0.0638 (0.0417)	0.0707 (0.0571)	0.0634 (0.0442)	0.0568 (0.0399)
RMSE	**0.0462**	0.0574	0.0652	0.0523	0.0586
			Student-t Distribution		
MSE	0.0814 (0.0728)	0.0635 (0.0711)	0.0692 (0.0145)	**0.0562** (0.0028)	0.0584 (0.0049)
MAE	0.0213 (0.0582)	0.0214 (0.0649)	0.0118 (0.0092)	**0.0093** (0.0036)	0.0115 (0.0013)
RMSE	0.2852	0.2519	0.2613	**0.2371**	0.2417
			Contaminated Normal Distribution		
MSE	0.0428 (0.0852)	0.0693 (0.0835)	0.0469 (0.0264)	**0.0381** (0.0152)	0.0401 (0.0173)
MAE	0.0209 (0.0369)	0.0213 (0.0492)	0.0117 (0.0329)	**0.0092** (0.0163)	0.0114 (0.0196)
RMSE	0.2069	0.2633	0.2166	**0.1952**	0.2003
			Skewed-t Distribution		
MSE	0.0077 (0.0109)	0.0049 (0.0074)	0.0068 (0.0154)	**0.0047** (0.0097)	0.0233 (0.0168)
MAE	0.0665 (0.0513)	0.0513 (0.0409)	0.0587 (0.0526)	**0.0492** (0.0412)	0.0609 (0.1374)
RMSE	0.0876	0.0700	0.0826	**0.0686**	0.1525

Note: S.E. in the parenthesis. Bold values show minimum value for each row.

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4.2 Empirical Analysis

The daily closing prices of Karachi Stock Exchange (KSE100 index) are utilized in this empirical analysis. The data are obtained from the International Monetary Fund (IMF) website from 3rd January 2007 to 31st December 2016, a total of *n=2424* observations. This time span includes the high and low volatile periods.

The log-returns at time t are defined as $r_t = 100 \times (\log(p_t) - \log(p_{t-1}))$ where $t = 1,2,3 \ldots, n$ and p_t denote the daily closing prices of KSE100 index at time *t*. The whole data set is divided into two parts. The in-sample part consists of initial *N=2000* observations which is used for the estimation of GARCH-type models and the out-sample or validation part consists of $K=n-N$ observations ($K=424$). The out-of-sample part is used to obtain one-day-ahead forecasts of VaR using recursive scheme.

The daily prices of KSE100 index along with log-returns are illustrated in Figure 1. Volatility clustering can be observed with the existence of high and low volatile periods. Therefore, main interest lies in modelling and forecasting volatility and risk of KSE during this particular period.

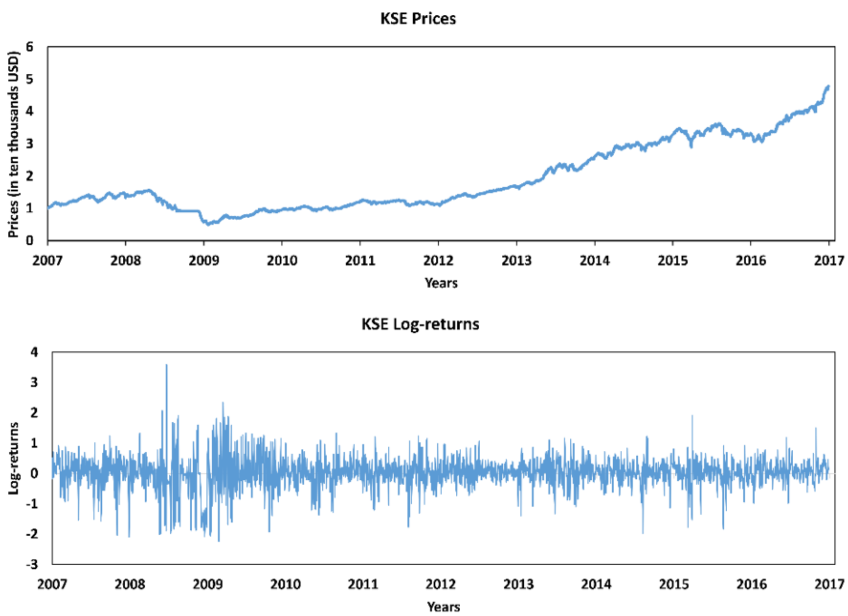

Figure 1. Daily Closing Prices and Log-returns of KSE

Table 3 presents the descriptive summary statistics of KSE100 log-returns. The mean log-returns are close to zero and the data are found to be negatively skewed with excess kurtosis.

At 5% significance level, the Jarque-Bera (JB) test rejects the normality of log-returns and the Ljung-Box test on squared log-returns at lag 20 is found highly significant indicating dependence in the squared log-returns. Therefore, GARCH-type models may be appropriate to be fitted to this data set.

Table 3
Descriptive Summary of log-returns of KSE

Mean	0.0274
Median	0.0355
Standard Deviation	0.4996
Variance	0.2496
Kurtosis	4.2017
Skewness	-0.3745
Minimum	-2.2300
Maximum	3.5849
Sum	67.6605
Jarque-Bera	206.3639
$Q^2(20)$	162.9328
Total Observations(n)	2470

Note: Q^2 denotes the LB statistic of squared log-returns on lag 20.

Both symmetric (the GARCH (1,1))and asymmetric (the GJR (1,1)) models are fitted to the KSE data and parameters are estimated using various M-estimators. The estimated parameters along with their corresponding standard errors of the GARCH (1,1) model are reported in Table4.The estimated parameters based on various score functions are found significant at 5% level. Also note, that the estimates of $\beta_2$have almost similar value free from c_H. TheLjung-Box test at lag 20 on squared residualsat 5% significance level is not found significant for all estimators, demonstrating the adequacy of fitting the GARCH(1, 1) model.

Table 4
M-estimates for GARCH (1, 1) parameters for KSE data

Parameters	QMLE	LAD	B estimator	ST	AL
$c_H\beta_0$	0.0102	0.0050	0.0068	0.0209	0.0095
	(0.0033)	(0.0014)	(0.0019)	(0.0030)	(0.0053)
$c_H\beta_1$	0.1138	0.0774	0.1173	0.1339	0.1495
	(0.0253)	(0.0128)	(0.0202)	(0.0244)	(0.0506)
β_2	0.8346	0.8276	0.8369	0.8461	0.8245
	(0.0351)	(0.0281)	(0.0352)	(0.0224)	(0.0559)
$Q^2(20)$	13.5117	12.6611	12.5791	12.5173	12.4266
p-value	0.8871	0.8814	0.8846	0.8944	0.8847

Note: S.E. in the parenthesis. Q^2 denotes the LB statistic for squared residuals on lag 20.

The estimated parameters of the GJR (1,1) model with their standard errors are shown in Table 5. Again, all estimates are found significant and the estimates of β_2 are close to each other. The Ljung-Box statistic and corresponding p-values are also reported. High p-values of the Ljung-Boxtest on squared residuals at lag 20 suggest the adequacy of the GJR (1,1) model for KSE data.

Table 5
M-estimates for GJR (1, 1) model for KSE data

Parameters	QMLE	LAD	B estimator	ST	AL
$c_H\beta_0$	0.0095	0.0066	0.0086	0.0214	0.0123
	(0.0001)	(0.0002)	(0.0001)	(0.0002)	(0.0001)
$c_H\beta_1$	0.0427	0.0405	0.0680	0.0729	0.0768
	(0.0002)	(0.0002)	(0.0006)	(0.0060)	(0.0006)
β_2	0.8383	0.8192	0.8397	0.8292	0.8190
	(0.0013)	(0.0011)	(0.0014)	(0.0012)	(0.0010)
$c_H\beta_3$	0.1956	0.1161	0.1518	0.2958	0.2193
	(0.0015)	(0.0010)	(0.0031)	(0.0040)	(0.0034)
$Q^2(20)$	9.7408	10.9618	13.7241	12.5074	10.1683
p-value	0.9726	0.9472	0.8442	0.8975	0.9650

Note: S.E. in the parenthesis. Q^2 denotes the LB statistic for squared residuals on lag 20.

Next, for the evaluation of the VaR estimates, we present the results of various back testing measures and tests. These methods are used to evaluate the accuracy of the in-sample and reliability of out-of-sample VaR estimates.

In-sample evaluation of VaR

Data from 3rd January 2007 to 31st December 2016 (2000 observations), considered as the in-sample period,are utilized for in-sample VaR evaluation. Table6 reports the results for the various backtesting methods for the GARCH (1, 1) model for p = 1%, 5% and 10%. The first row of the table shows that, for all estimators, the estimated value$\hat{p}$'s is close to the corresponding value of p.At $p = 1\%$, both the unconditional coverage(LR_{uc}) and conditional coverage (LR_{cc}) statistics are not found significant at 5% level indicating that the expected and actual proportion of observations falling below the VaR threshold are similar. The DQ test for higher order dependence is rejected for all the estimators and the mean relative biases do not have much variations. However, the LAD estimator showed the minimum value of MRB. The

ST estimator has the smallest value of AQL which indicates the better performance of this estimator than others in terms of accurate VAR estimates.

Table 6
In-sample evaluation of VaR for GARCH model

	QMLE	LAD	B-estimator	ST	AL
			99% VaR Confidence Level		
$\hat{p}$	0.0100	0.0100	0.0100	0.0100	0.0100
LR_{uc}	0	0	0	0	0
LR_{cc}	0.4244	0.4244	0.4244	0.4244	0.4244
DQ	6.0940	7.3448	13.2640	7.3005	7.3259
MRB	0.0217	0.0019	-0.0177	0.0129	0.0024
AQL	0.0123	0.0123	0.0124	**0.0120**	0.0123
			95% VaR Confidence Level		
$\hat{p}$	0.0485	0.0485	0.0495	0.0490	0.0480
LR_{uc}	0.0956	0.0956	0.0106	0.0424	0.1706
LR_{cc}	1.2831	0.5590	0.3674	1.1279	0.6947
DQ	18.0380*	12.2843	9.6292	14.6902*	13.2125
MRB	0.0174	0.0090	-0.0215	0.0078	0.0098
AQL	0.0600	0.0598	0.0615	0.0608	**0.0593**
			90% VaR Confidence Level		
$\hat{p}$	0.0990	0.0970	0.0980	0.0990	0.0970
LR_{uc}	0.0223	0.2018	0.0894	0.0223	0.2018
LR_{cc}	29.9232*	21.9482*	17.1053*	21.5782*	21.9482*
DQ	65.6948**	49.3977**	38.5342**	51.0328**	49.3981**
MRB	0.0003	0.0014	0.0012	-0.0082	-0.0001
AQL	0.1284	0.1248	0.1248	0.1282	**0.1247**

Note: DQ: Dynamic Quantile statistic; * and ** denotes significance at 5% and 1% levels, respectively; MRB= Mean Relative Bias, AQL= Average Quadratic loss.

In case of $p = 5\%$, again LR_{uc} and LR_{cc} statistics are not found significant. The DQ test for no higher order dependence is accepted for all estimators except QMLE and ST estimator. Moreover, results also indicate that AL estimator produced the lowest MRB and AQL values than competing estimators. For $p = 10\%$ (90% VaR confidence level) both the coverage statistic, LR_{uc} and LR_{cc} are found to have different results. The LR_{cc} is rejected for all estimators while the LR_{uc} is found not significant. Also, the DQ test for all estimators indicate the higher order dependence in the VaR estimates. Overall, the AL estimator gives better performance in terms of lowest MRB and AQL.

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In Table7, the in-sample VaR estimates results for the GJR (1, 1) model are reported. Similar to the results of GARCH (1, 1) model,$\hat{p}$'sare found close to the expected value of p.At 99% VaR level, both coverage statistics are not significant and the DQ test is rejected for all the estimator. The MRB and AQL values are found smallest for the LAD and ST estimator, respectively. For 95% VaR level, the ST and AL estimators produced the smallest MRB and the AQL, respectively. Finally, for 90% VaR confidence level, theLR_{uc}, $LR_{cc}$and DQ tests are found to be non-significant for all estimators except for the QMLE. Again, we noted the superior performance of the AL estimator in terms of smallest AQL and MRB. Therefore, the overall in-sample VaR evaluation results reveal that both ST and AL estimators provide better results for KSE100 Index and may be preferred over frequently used estimators for VaR.

Table 7

In-sample evaluation of VaR for GJR model

	QMLE	LAD	B-estimator	ST	AL
			99% VaR Confidence Level		
$\hat{p}$	0.0095	0.0095	0.0095	0.0095	0.0095
LR_{uc}	0.0514	0.0514	0.0514	0.0514	0.0514
LR_{cc}	0.4351	0.4351	0.4351	0.4351	0.4351
DQ	6.4629	6.6804	7.8439	6.3749	6.6670
MRB	-0.0123	-0.0013	0.0024	0.0064	-0.0026
AQL	0.0123	0.0124	0.0126	**0.0123**	0.0124
			95% VaR Confidence Level		
$\hat{p}$	0.0480	0.0480	0.0465	0.0455	0.0480
LR_{uc}	0.1706	0.1706	0.5276	0.8780	0.1706
LR_{cc}	0.3619	0.9694	1.1196	1.3507	0.9694
DQ	5.1190	3.1735	2.5667	3.0820	3.1641
MRB	0.0146	-0.0024	0.0014	0.0020	-0.0019
AQL	0.0601	0.0601	0.0585	**0.0578**	0.0601
			90% VaR Confidence Level		
$\hat{p}$	0.0960	0.0960	0.0970	0.0955	0.0945
LR_{uc}	0.3599	0.3599	0.2018	0.4561	0.6835
LR_{cc}	9.6063*	2.4694	2.0372	2.0757	3.2466
DQ	24.6727**	9.3620	8.7354	10.1566	11.0926
MRB	-0.0023	0.0020	0.0010	-0.0005	0.0027
AQL	0.1223	0.1206	0.1211	0.1205	**0.1190**

Note: DQ: Dynamic Quantile statistic where * and ** denotes significance at 5% and 1% levels, respectively; MRB= Mean Relative Bias, AQL= Average Quadratic loss.

Out-of-sample evaluation of VaR

In this subsection, the performance of M-estimators is evaluated for one-step-ahead forecasts of VaR for KSE data. The out-of-sample predictions are valuable for risk and portfolio management and also for investors who desire to measure the model's performances on basis of risk forecasts. The results at different VaR confidence levels are reported in Table 8 for the GARCH(1, 1) model. First, we noted that $\hat{p}$values are close to the corresponding rejection probabilities for all estimators except the QMLE. Both coverage statistics,LR_{uc} and $LR_{cc}$are found non-significant at 5% significance level which reveals that proportions of violations are approximately equal to the expected proportions for 99% and 95% VaR confidence levels. The conditional coverage tests was rejected for 99% VaR level. The null hypothesis of no higher order dependence in VaR violations cannot be rejected using the DQ test at all VaR confidence levels. The AL estimator produces minimum MRB whereas the ST estimator is found to show the least AQL.

Table 8
Out-of-sample evaluation of VaR for GARCH model

	QMLE	LAD	B-estimator	ST	AL
			90% VaR Confidence Level		
$\hat{p}$	0.0047	0.0071	0.0094	0.0047	0.0071
LR_{uc}	1.4863	0.4080	0.0140	1.4863	0.4080
LR_{cc}	1.5147	04650	0.1093	1.5147	0.4650
DQ	1.5082	0.4918	0.2073	1.5382	0.4811
MRB	0.0229	-0.0046	-0.0182	0.0284	-0.0021
AQL	0.0053	0.0077	0.0101	**0.0052**	0.0077
			95% VaR Confidence Leve		
$\hat{p}$	0.0354	0.0377	0.0377	0.0354	0.0377
LR_{uc}	2.1164	1.4616	1.4616	2.1164	1.4616
LR_{cc}	2.5461	1.7743	1.7743	2.5461	1.7743
DQ	3.8006	3.8849	4.0229	3.7851	3.8837
MRB	0.0358	-0.0100	-0.0281	0.0296	0.0004
AQL	0.0410	0.0437	0.0440	**0.0410**	0.0437
			90% VaR Confidence Level		
$\hat{p}$	0.0755	0.0755	0.0802	0.0731	0.0755
LR_{uc}	3.0705	3.0705	1.9700	3.7212	3.0705
LR_{cc}	7.8999*	7.8999*	5.7776*	9.1214*	7.8999*
DQ	10.1665	10.7410	9.8956	11.7510	10.7138
MRB	0.0157	-0.0004	-0.0130	0.0125	0.0001
AQL	0.0877	0.0877	0.0926	**0.0852**	0.0877

Note: DQ: Dynamic Quantile statistic; * and ** denotes significance at

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5% and 1% levels, respectively; MRB= Mean Relative Bias;AQL= Average Quadratic loss.

The out-of-sample VaR estimates from the GJR(1, 1) model are reported in Table9.Results similar to the out-of-sample VaR evaluation of the GARCH(1, 1) model are observed with estimators such as LAD, ST and AL providing better forecasts for one day ahead VaR.

Table 9
Out-of-sample evaluation of VaR for GJR model

	QMLE	LAD	B-estimator	ST	AL
			99% VaR Confidence Level		
$\hat{p}$	0.0047	0.0071	0.0071	0.0071	0.0071
LR_{uc}	1.4863	0.4080	0.4080	0.4080	0.4080
LR_{cc}	1.5147	0.4650	0.4650	0.4650	0.4650
DQ	2.5913	1.7279	1.6913	1.7279	1.7279
MRB	0.0147	-0.0096	-0.0015	-0.0096	-0.0096
AQL	0.0081	0.0076	0.0074	0.0074	**0.0073**
			95% VaR Confidence Level		
$\hat{p}$	0.0330	0.0330	0.0354	0.0330	0.0330
LR_{uc}	2.9095	2.9095	2.1164	2.9095	2.9095
LR_{cc}	3.4258	3.4858	2.5461	3.4258	3.4258
DQ	4.3147	4.2478	3.7423	4.2478	4.2478
MRB	0.0388	-0.0056	-0.0184	-0.0056	-0.0056
AQL	0.0384	0.0383	0.0407	0.0383	**0.0379**
			90% VaR Confidence Level		
$\hat{p}$	0.0660	0.0684	0.0684	0.0684	0.0684
LR_{uc}	6.0998*	5.2337*	5.2337*	5.2337*	5.2337*
LR_{cc}	6.9386*	7.2629*	5.8998*	7.2629*	7.2629*
DQ	7.2213	7.9970	7.1286	7.9970	7.9970
MRB	0.0204	-0.0046	-0.0085	-0.0046	-0.0046
AQL	0.0864	0.0791	**0.0790**	0.0791	0.0791

Note: DQ: Dynamic Quantile statistic; * and ** denotes significance at 5% and 1% levels, respectively; MRB= Mean Relative Bias; AQL= Average Quadratic loss.

Overall, the performance of proposed Skewed-*t* and Asymmetric Laplace estimators are found superior than other estimators understudy for the KSE data whereas the study of

Iqbal(2017) reveals B-estimator to be a better fit for this data. Analyses of M-estimators indicate the need for using robust estimators ST, AL, LAD and B-estimator for estimation and prediction of VaR than the commonly used QMLE.

5 CONCLUSIONS

In the article, two robust estimators based on the score functions obtained from Skewed-*t* and Asymmetric Laplace distributions are proposed for GARCH-type models. Results of simulation study revealed the superior performance of these estimators over other competing estimators like QMLE, LAD and B-estimator in terms of providing accurate estimates of parameters when errors were generated form heavy-tailed, contaminated and skewed distributions. These estimators were also applied in an empirical application to daily log-returns of Karachi Stock Exchange. The accuracy and relative performance of all estimators under study, in estimating and predicting VaR, were evaluated through various backtesting approaches. The findings of empirical analysis highlighted that the ST and AL estimators provide better estimates of both in-sample VaR and the out-of-sample VaR. Therefore, it is suggested that for accurate estimation of GARCH-type models and better forecasts of VaR, these estimators may be used especially for non-normal.

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